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Comparison of Solutions from Parabolic and Hyperbolic Models for Transient Heat Conduction in Semi-Infinite Medium

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Abstract The expression for the transient temperature during damped wave conduction and relaxation developed by Baumeister and Hamill by the method of Laplace transforms was further integrated. A Chebyshev polynomial approximation was used for the integrand with a modified Bessel composite function in space and time. A telescoping power series leads to a more useful expression for the transient temperature. By the method of relativistic transformation, the transient temperature during damped wave conduction and relaxation was developed. There are four regimes to the solution. These include: (i) a regime comprising a Bessel composite function in space and time, (ii) another regime comprising a modified Bessel composite function in space and time, (iii) the temperature solution at the wave front was also developed separately, and (iv) the fourth regime at a given location X in the medium is at times less than the inertial thermal lag time. In this regime, the temperature was found to be unchanged at the initial condition. The solution for the transient temperature from the method of relativistic transformation is compared side by side with the solution for the transient temperature from the method of Chebyshev economization. Both solutions are within 12 % of each other. For conditions close to the wave front, the solution from the Chebyshev economization is expected to be close to the exact solution and was found to be within 2 % of the solution from the method of relativistic transformation. Far from the wave front, i.e., close to the surface, the numerical error from the method of Chebyshev economization is expected to be significant and verified by a specific example. The solution for transient surface heat flux from the parabolic Fourier heart conduction model and the hyperbolic damped wave conduction and relaxation models are compared with each other. For $\tau > 1/2$ the parabolic and hyperbolic solutions

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are within 10 % of each other. The parabolic model has a "blow-up" as $\tau \rightarrow 0$, and the hyperbolic model is devoid of singularities. The transient temperature from the Chebyshev economization is within an average of 25 % of the error function solution for the parabolic Fourier heat conduction model. A penetration distance beyond which there is no effect of the step change in the boundary is predicted using the relativistic transformation model.

Keywords Damped wave conduction and relaxation · Laplace transforms · Microscale heat transfer · Parabolic and hyperbolic models · Relativistic transformation

Nomenclature

erf (r)	Error function $\mathrm{erf}(r) = \frac{2}{\sqrt{\pi}} \int_0^r \mathrm{e}^{-r^2} \mathrm{d}r$
$H(\tau)$	Space integrated temperature
$I_0(x)$	Modified Bessel function of the first kind and zeroth order
$I_1(x)$	Modified Bessel function of the first kind and first order
$J_0(x)$	Bessel function of the first kind and zeroth order
k	Thermal conductivity of semi-infinite medium ($\mathrm{W \cdot m^{-1} \cdot K^{-1}}$)
$K_0(x)$	Modified Bessel function of the second kind and zeroth order
m	Power series index
p	Integration variable
q	Heat flux ($\mathrm{W \cdot m^{-2}}$)
q^*	Dimensionless heat flux $q^* = \frac{q\sqrt{\tau_r}}{(T_s - T_0)\sqrt{k\rho C_p}}$
r	Change of variable for interval (−1, 1) $r = \frac{(2\psi - \tau - X)}{(\tau - X)}$
s	Laplace transform variable $\mathrm{L} f(t) = \int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t \ldots t \geq 0 f(t)$
T_0	Initial temperature (K)
T_s	Surface temperature (K)
$T_n(r)$	Chebyshev polynomial (Tables 1, 2)
u	Dimensionless temperature $\left(u = \frac{(T - T_0)}{(T_s - T_0)}\right)$
X	Dimensionless distance $X = \frac{x}{\sqrt{\alpha\tau_r}}$
$Y_0(x)$	Bessel function of the second kind and zeroth order

Greek

α	Thermal diffusivity of semi-infinite medium ($\mathrm{m^2 \cdot s^{-1}}$)
η	Wave transformation, $\eta = \tau + X$
ξ	Wave transformation, $\xi = \tau - X$
ρ	Density of semi-infinite medium ($\mathrm{kg \cdot m^{-3}}$)
τ	Dimensionless time (t/τ_r)
τ_r	Relaxation time (s)
θ	$\cos^{-1}(r)$
ψ	Transformation variable, $\psi = p^2 - X^2$

1 Introduction

A generalized Fourier's law of heat conduction was sought for six reasons [1]. Boley [2] showed that the addition of the second derivative in time of temperature to the governing equation is mathematically the only way to remove singularities found in the solution to parabolic heat conduction equations. Reference to the generalized Fourier's law of heat conduction can be traced back to Maxwell [3], Morse and Feshbach [4], Cattaneo [5], and Vernotte [6] postulated this equation independently. This equation can be used to account for a finite speed of heat. Reviews in the literature of the use of this equation have been provided by Joseph and Preziosi [7] and Ozisik and Tzou [8]. Tzou [9] has discussed the micro- to macro-scale behavior. Sharma [10–13] discussed the manifestation of the damped wave transport and relaxation equation in industrial applications and provided bounded solutions within the constraints of the second law of thermodynamics. It was shown that the generalized Fourier's law of heat conduction can be derived by including the acceleration term in the free electron theory, the acceleration term in the Stokes–Einstein theory for molecular diffusion, by accounting for the accumulation term in the kinetic theory of gases, and combining in series the Hooke's elastic element and Newton's viscous element in the viscoelastic theory. The relaxation time was found to be a third of the collision time of the electron and the obstacle. The velocity of heat was found to be identical with the velocity of mass derived from the kinetic representation of pressure or the Maxwell representation of the speed of molecules. Some investigators [14] have used the Boltzmann transport equation and derived both the Fourier's law of heat conduction and the damped wave conduction and relaxation equation as special cases. They derive a set of equations for length scales comparable to the mean free path of the molecule. Ali [15,16] used statistical mechanics and the kinetic theory and derived the generalized Fourier's law of heat conduction for monatomic and diatomic gases. Glass and McRae [17] looked at the variable specific heat and thermal relaxation parameter. Luikov [18] discussed the hyperbolic heat conduction equation. He provided a range of relaxation times from milliseconds to 1000 s. Both heat transfer in turbulent systems and heat conduction in metals can be represented using the same relaxation parameter.

The relaxation time has been measured by Brown and Churchill [19], Peshkov [20], and Zehnder and Rosakis [21]. The relaxation mechanism is fundamental to thermal resonance that cannot be depicted by Fourier's law of heat conduction [22]. For a thermal wave speed around 900 $m \cdot s^{-1}$ in 4340 steel at 480 °C, the value of the relaxation time was found to be of the order of 10^{-11} s. A table for the relaxation times for different materials at different temperatures and pressures is not available in the literature. Relaxation times for materials with a nonhomogeneous inner structure were presented by Kaminski [23]. For sodium bicarbonate they report a relaxation time of 29 s, 20 s for sand, and 54 s for ion exchange materials. Mitura et al. [24] claim that for the falling drying rate period the average time is of the order of several thousand seconds. For homogeneous substances the relaxation time values range from 10^{-8} s to 10^{-10} s; for gases to 10^{-10} s to 10^{-12} s; for liquids and dielectric solids as concluded by Sieniutycz [25]. Mitra et al. [26] presented experimental evidence of the wave nature of heat propagation in processed meat and demonstrated that the hyperbolic heat conduction model is an accurate representation on a macroscopic level of the heat conduction process in such a biological material. They report a relaxation time of the order of 16 s.

Some investigators have raised some concerns about violations of the second law of thermodynamics by the hyberbolic heat conduction equation. Bai and Lavine [27], Taitel [28], Zanchini [29], and Barletta and Zanchini [30]. Taitel [28] attempted to obtain an analytical solution to the governing equation and found that the solution temperature for some values exceeded the boundary temperature indicating a possible violation of the Clausius inequality. Al Nimr et al. [31] discuss the 'temperature overshoot' phenomena. Haji Sheik et al. [32] pointed out some anomalies in the hyperbolic heat equation. Al Nimr et al. [33,34] investigated nonequilibrium entropy production under the effect of the dual-phase lag heat conduction model. They found that the entropy production cannot be described using the classical form of the equilibrium entropy production where using this form leads to the violation of the second law of thermodynamics. Transient instability, including the intrinsic transition from the desirable stability, neutral stability to the ultimate unstable response was investigated by Tzou [35] for a wide spectrum of heating rates. Tzou confirmed that the relaxation time results from the rate equation within the mainframe of the second law in nonequilibrium, irreversible thermodynamics. Schnaid [36] attempted to derive the governing equation for heat conduction with a finite speed of heat propagation directly from classical thermodynamics. Cai et al. [37] presented algebraically explicit analytical solutions of hyperbolic type heat conduction equations in three dimensions. Lin and Chen [38] sought numerical solutions of hyperbolic heat conduction in cylindrical and spherical systems. Antaki [39] examined the dual-phase lag equation that was introduced by Tzou and provided an analytical solution for the case of a semi-infinite medium subject to constant wall flux boundary condition. Lewandowsha and Malinowski [40] attempted to provide an analytical solution of the hyperbolic heat conduction equation for the case of a finite medium symmetrically heated on both sides using the method of Laplace transforms. Volz et al. [41] used a molecular dynamics numerical solution to test the validity of the generalized Fourier's law of heat conduction. They confirmed the generalized law when considering heat flux fluctuations at equilibrium. Temperature overshoot and undershoot were discussed by Tan and Yang [42] during thermal propagation of thermal waves in a thin film under transient conditions. Tian [43] mentioned that the basic waveform of thermal waves is hyperbolic waves.

Sharma [1] presented an analytical solution for the case of a finite slab subject to a constant wall temperature. The final condition in time as the fourth condition for the second-order hyperbolic PDE governing equation was shown to result in well-bounded solutions. This indicates that care must be taken for the choice of the conditions used in the boundaries of space and initial and final time values. They have to be physically reasonable. For example, at steady state, an equilibrium temperature is attained. Only for large relaxation times oscillations were found in the solution for temperature. These oscillations were found to be sub-critical and damped. The time conditions used by Taitel are unrealistic from the physical realities of heat transfer. That may be the reason that their solution exhibited a temperature overshoot. Thus, the equations do not violate the laws of thermodynamics as much as the choice of space and time conditions as constraints. Sharma [10] also showed that a temperature undershoot can occur when Fourier's law of heat conduction is applied at a steady state in the presence of a temperature-dependent heat source. This is in violation of the third law of thermodynamics. Here again, the choice of the space condition at some arbitrary length is not sufficient. The critical length beyond which no heat transfer will occur will have to be identified to keep the solution from violating the third law of thermodynamics.

Baumeister and Hamill [44] presented an analytical solution for the transient temperature during damped wave conduction and relaxation by the method of Laplace transforms for a semi-infinite medium subject to a constant wall temperature boundary condition. Their solution is for the open interval, $\tau > X$. The solution is in the form of an integral, where the integrand is a modified Bessel composite function in space and time of the first order and first kind. Their solution exhibits a sharp discontinuity at the wave front. Barletta and Zanchini [30] found the solution of the hyperbolic heat conduction equation in a semi-infinite medium to be in violation of the second law of thermodynamics for cylindrical domains. Although there are a number of studies discussing parabolic, hyperbolic models, there is little work done on where and when and by how much these models differ and where and why they will be the same. Sharma [1] proposed a novel transformation to obtain an analytical solution for the damped wave conduction and relaxation equation by the method of relativistic transformation of coordinates. The solution in Baumeister and Hamill [44] is simplified into a useful expression using a Chebyshev polynomial approximation in this study. The solution is compared with the solution from the method of relativistic transformation of the hyperbolic damped wave conduction and relaxation equation quantitatively as well as qualitatively. The parabolic and hyperbolic solutions are compared with each other for the surface flux. It is shown that bounded, co-continuous analytical solutions can be obtained for the generalized Fourier's law of heat conduction by using the method of relativistic transformation of coordinates.

2 Theory

2.1 Parabolic Versus Hyperbolic

Consider a semi-infinite medium at an initial temperature of T_0 (Fig. 1). For times greater than 0, the surface at $x = 0$ is maintained at a constant surface temperature

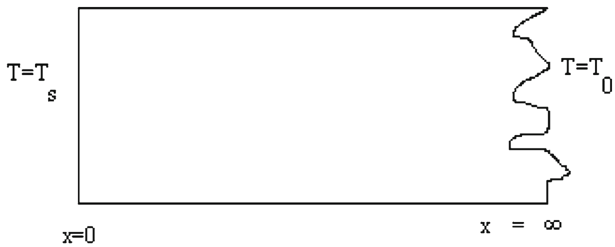

Fig. 1 Semi-infinite medium with initial temperature at T_0

at $T = T_{\mathrm{s}}$, $T_{\mathrm{s}} > T_0$. The boundary and initial conditions are as follows:

$$t = 0, \quad T = T_0 \tag{1}$$

$$x = 0, \quad T = T_{\mathrm{s}} \tag{2}$$

$$x = \infty, \quad T = T_0 \tag{3}$$

The transient temperature in the semi-infinite medium can be solved using the Fourier parabolic heat conduction equations and the Boltzmann transformation, $\eta = \frac{x}{\sqrt{4\alpha t}}$ and shown to be [45]

$$u = \frac{(T - T_0)}{(T_{\mathrm{s}} - T_0)} = 1 - \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right) \tag{4}$$

The heat flux can be written as

$$q^* = \frac{q}{\sqrt{k\rho C_p/\tau_{\mathrm{r}}}\,(T_{\mathrm{s}} - T_0)} = \frac{1}{\sqrt{\pi\tau}} \exp\left(-\frac{x^2}{4\alpha t}\right) \tag{5}$$

The dimensionless heat flux at the surface is then given by

$$q_{\mathrm{s}}^* = \frac{1}{\sqrt{\pi\tau}} \tag{6}$$

It can be seen that there is a "blow-up" in Eq. 6 as $\tau \to 0$. For applications of substantial industrial importance such as the heat transfer between fluidized beds to immersed surfaces [10], there have been found large deviations between experimental data and mathematical models based upon surface renewal theory. The critical parameter in the mathematical models is the contact time of the packets that are composed of solid particles at the surface. This contact time is small for gas–solid fluidized beds for certain powder types. Under such circumstances, the microscale time effects may have been significant. These are not accounted for by the parabolic heat conduction models. This is one of the motivations for studying the hyperbolic heat conduction models. Boley [2] has shown that the ballistic term in the governing hyperbolic heat conduction equation is the "only" mathematical modification to the parabolic heat conduction equation that can remove the singularity in Eq. 6 at short times.

The governing hyperbolic heat conduction equation in one dimension for a semi-infinite medium with constant thermophysical properties, ρ, C_p, k, and τ_r, i.e., the density, heat capacity, thermal conductivity, and thermal relaxation time, respectively, can be obtained by combining the damped wave conduction and relaxation equation with the energy balance equation to yield

$$\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial X^2} \tag{7}$$

where

$$u = \frac{(T - T_0)}{(T_s - T_0)}; \quad X = \frac{x}{\sqrt{\alpha \tau_r}}; \quad \tau = \frac{t}{\tau_r} \tag{8}$$

This is the damped wave conduction and relaxation equation. When the relaxation time is zero, Eq. 7 will revert to the parabolic partial differential equation, PDE, for transient conduction from Fourier's law. When the rate of change of the temperature with time is much greater than an exponential rise with time, e^{τ}, Eq. 7 will revert to the wave equation (Tzou [9] and Sharma [10]). Equation 7 is a hyperbolic partial differential equation, which is second order with respect to space and second order with respect to time.

Baumeister and Hamill [44] obtained the Laplace transform of Eq. 7 and applied the boundary condition at ∞, given by Eqs. 2 and 3 to obtain in the Laplace domain,

$$\bar{u} = \frac{\exp\left(-X\sqrt{s\,(s+1)}\right)}{s} \tag{9}$$

They used the initial time conditions of

$$t = 0, \quad u = 0 \tag{10}$$

$$t = 0, \quad \frac{\partial u}{\partial \tau} = 0 \tag{11}$$

They integrated Eq. 9 with respect to space to obtain

$$H(s) = \int \exp \frac{-X\sqrt{s(s+1)}}{s}\, \mathrm{d}X = -\frac{1}{s\sqrt{s(s+1)}} \exp \frac{-X\sqrt{s(s+1)}}{s} \tag{12}$$

The inversion of Eq. 12 was obtained from the Laplace transform tables [10] and found to be

$$H(\tau) = \int_0^{\tau} \exp\left(-\frac{p}{2}\right) I_0\left(\frac{\sqrt{p^2 - X^2}}{2}\right) \mathrm{d}p \tag{13}$$

The dimensionless temperature is obtained by differentiating $H(\tau)$ in Eq. 11 with respect to X, and for $\tau \geq X$,

$$u = \frac{\partial H}{\partial X} = -X \int_X^{\tau} \exp\left(-\frac{p}{2}\right) \frac{I_1(0.5\sqrt{p^2 - X^2})}{\sqrt{p^2 - X^2}}\, \mathrm{d}p + \exp\left(-\frac{X}{2}\right) \tag{14}$$

Baumeister and Hamill [44] presented their solution in integral form as shown in Eq. 14. In this study, the integrand is approximated to a Chebyshev polynomial, and a useful expression for the dimensionless temperature is obtained. This is used to compare the results with those obtained by a relativistic transformation (Sharma [10]). The dimensionless heat flux can be seen to be

$$q^* = \exp\left(-\frac{\tau}{2}\right) I_0\left(\frac{\sqrt{\tau^2 - X^2}}{2}\right) \tag{15}$$

The surface heat flux can be seen to be

$$q_s^* = \exp\left(-\frac{\tau}{2}\right) I_0\left[\frac{\tau}{2}\right] \tag{16}$$

2.2 Chebyshev Economization Telescoping Power Series

In order to further study the dimensionless transient temperature from the hyperbolic damped wave conduction and relaxation equation, the integral expression given by Baumeister and Hamill [44] in Eq. 14 can be simplified using a Chebyshev polynomial [46]. Chebyshev polynomial approximations tend to distribute the errors more evenly with a reduced maximum error by use of the cosine functions. The set of polynomials, $T_n(r) = \cos(n\theta)$ generated from the sequence of cosine functions using the transformation,

$$\theta = \cos^{-1}(r) \tag{17}$$

represent Chebyshev polynomials (Table 1). Coefficients of the Chebyshev polynomials for the integrand in Eq. 12, $\frac{I_1 1/2\sqrt{p^2-X^2}}{\sqrt{p^2-X^2}}$ can be computed with some effort. The modified Bessel function of the first order and first kind can be expressed as a power series as follows:

$$\frac{I_1 1/2\sqrt{p^2 - X^2}}{\sqrt{p^2 - X^2}} = \sum_{m=0}^{\infty} \frac{\left(p^2 - X^2\right)^m}{4^{2k+1}(m!)(m+1)!} = \sum_{m=0}^{\infty} \frac{\psi^m}{4^{2k+1}(m!)(m+1)!} \tag{18}$$

where $\psi = p^2 - X^2$.

Table 1 Chebyshev polynomials

$T_0(r) = 1$
$T_1(r) = 2$
$T_2(r) = 2r^2 - 1$
$T_3(r) = 4r^3 - 3r$
$T_4(r) = 8r^4 - 8r^2 + 1$
$T_5(r) = 16r^5 - 20r^3 + 5r$
$T_6(r) = 32r^6 - 48r^4 + 18r^2 - 1$

Table 2 Powers in terms of Chebychev polynomials

$1 = T_0(r)$
$r = T_1(r)$
$r^2 = \frac{1}{2}(T_0(r) + T_2(r))$
$r^3 = \frac{1}{4}(3T_1(r) + T_3(r))$
$r^4 = \frac{1}{8}(3T_0(r) + 4T_2(r) + T_4(r))$
$r^5 = \frac{1}{16}(10T_1(r) + 5T_3(r) + T_5(r))$
$r^6 = \frac{1}{32}(10T_0(r) + 15T_2(r) + 6T_4(r) + T_6(r))$

Each of the ψ^m terms can be replaced with its expansion in terms of Chebyshev polynomials as given in Table 2.

The coefficients of like polynomials $T_i(r)$ are collected. When the truncated power series polynomial of the integrand (Eq. 18) is represented by a Chebyshev polynomial, some of the high-order Chebyshev polynomials can be dropped with negligible truncation error. This is because the upper bound for $T_n(r)$ in the interval $(-1, 1)$ is 1. The truncated series can then be re-transformed to a polynomial in r with fewer terms than the original and with modified coefficients. This procedure is referred to as Chebyshev economization or telescoping a power series.

Prior to expression of Eq. 18 in terms of Chebyshev polynomials, the interval (X, τ) needs to be converted to the interval $(-1, 1)$. So let,

$$r = \frac{2\psi - \tau - X}{\tau - X} \quad \text{and} \quad \psi = \frac{r(\tau - X) + (\tau + X}{2}, \tag{19}$$

Furthermore, let

$$\xi = (\tau - X) \quad \text{and} \quad \eta = (\tau + X) \tag{20}$$

Thus,

$$\psi = \frac{r\xi + \eta}{2} \tag{21}$$

Substituting Eq. 21 in Eq.18,

$$\frac{I_1\left(\frac{\sqrt{p^2-X^2}}{2}\right)}{\sqrt{p^2 - X^2}} = \sum_{m=0}^{\infty} \frac{(r\xi + \eta)^m}{2^k 4^{2k+1} m!\,(m+1)!} \tag{22}$$

The right-hand side (RHS) of Eq. 22 can be written as

$$\text{RHS Eq. 22} = \frac{1}{4} + \frac{r\xi + \eta}{256} + \frac{(r\xi + \eta)^2}{49,152} + \cdots \tag{23}$$

A truncation error of $\frac{(r\xi+\eta)^3}{18,874,368}$ is incurred in writing the LHS of Eq. 22 as Eq. 23.

Replacing the r, r^2, and r^3 terms in Eq. 23 in terms of Chebyshev polynomials given in Table 1 and collecting the like Chebyshev coefficients, T_0, T_1, and T_2, the RHS of Eq. 22 can be written as

$$\begin{aligned} T_0(r)\left(\frac{1}{4}+\frac{\eta}{256}+\frac{\eta^2}{49{,}152}+\frac{\xi^2}{98{,}304}\right) \\ +T_1(r)\left(\frac{\xi}{256}+\frac{2\eta\xi}{49{,}152}\right) \end{aligned} \tag{24}$$

The $T_2(r)$ term can be dropped with an added error of only $\frac{\xi^2}{98{,}304}$. The order of magnitude of the error incurred is thus, $O\left(\frac{\xi^2}{98{,}304}\right)$. Retransformation of the series given by Eq. 22 yields

$$\frac{I_1\left(\frac{\sqrt{p^2-X^2}}{2}\right)}{\sqrt{p^2-X^2}}=\frac{1}{4}-\frac{X^2}{128}+\frac{\eta^2}{49,152}+\frac{\xi^2}{98,304}+\frac{(p^2-X^2)}{128} \tag{25}$$

The error involved in writing Eq. 25 is $4.1\times10^{-5}\eta\xi$. If the Chebyshev polynomial approximation was not used for the integrand and the power series was truncated after the second term, the error would have been $4\times10^{-3}r^2$. Substituting Eq. 25 in Eq. 12 and further integrating the expression for the dimensionless temperature,

$$\begin{aligned} u=\exp\left(-\frac{X}{2}\right)+X\exp\left(-\frac{X}{2}\right)\left(\frac{5}{8}+\frac{X}{16}+\frac{\eta^2}{24{,}576}+\frac{\xi^2}{49.152}\right) \\ +X\exp\left(-\frac{\tau}{2}\right)\left(\frac{3}{8}-\frac{\tau}{16}-\frac{X^2}{64}+\frac{\eta^2}{24{,}576}+\frac{\xi^2}{49{,}152}\right) \end{aligned} \tag{26}$$

It can be seen that Eq. 26 can be expected to yield reliable predictions on the transient temperature close to the wave front. This is because the error increases as a function of $4.1\times10^{-5}\eta\xi$. Far from the wave front, i.e., close to the surface, the numerical error may become significant.

2.3 Method of Relativistic Transformation of Coordinates

Sharma [4] developed a relativistic transformation method to solve for the transient temperature by damped wave conduction and relaxation in a semi-infinite medium. The transient temperature was expressed as a product of a decaying exponential in time and wave temperature, i.e., $u = W\exp(-n\tau)$. This is typical of transient heat conduction applications. Also, the damping term in the hyperbolic PDE once removed will lead to an equation of the Klein–Gordon type that can be examined for the wave temperature without being clouded by the damping component. Let

$$u = W\exp(-n\tau) \tag{27}$$

The basis for this transformation is to recognize that the damped wave conduction and relaxation equation which is of the hyperbolic type has both a damping component and a wave component to it. In order to better study the characteristics of the wave component, it would be desirable to remove the damping component from the governing equation. The transformation given in Eq. 27 was selected to delineate the damping and wave components of the transient temperature. Furthermore, it is realized that transient temperatures decay with time exponentially. This leads to the negative exponent in the exponentiated term.

Then,

$$\frac{\partial u}{\partial \tau} = -n \exp(-n\tau)\, W + \exp(-n\tau) \frac{\partial W}{\partial \tau} \tag{28}$$

$$\frac{\partial^2 u}{\partial \tau^2} = +n^2 \exp(-n\tau)\, W - 2n \exp(-n\tau) \frac{\partial W}{\partial \tau} + \exp(-n\tau) \frac{\partial^2 W}{\partial \tau^2} \tag{29}$$

Equation 7 then becomes

$$\frac{\partial^2 W}{\partial X^2} = W\left(-n + n^2\right) + \frac{\partial W}{\partial \tau}(1 - 2n) + \frac{\partial^2 W}{\partial \tau^2} \tag{30}$$

It can be seen that at $n = 1/2$, the governing equation for temperature, Eq. 28, can be transformed as Eq. 31. At $n = 1/2$, it can be seen that the governing equation in the transient temperature reverts to a equation for the wave temperature. This happens to be a Bessel special differential equation;

$$\frac{\partial^2 W}{\partial \tau^2} - \frac{W}{4} = \frac{\partial^2 W}{\partial X^2} \tag{31}$$

This is the governing equation for the wave temperature, W. Once the damping component is removed as shown above, the characteristics of the wave temperature can be better studied. Equation 31 for the wave temperature can be transformed into a Bessel differential equation by the following substitution. Let

$$\psi = \tau^2 - X^2 \tag{32}$$

This substitution variable ψ can be seen to be a spatio-temporal variable. It is symmetric with respect to space and time. It is for the open interval, $\tau > X$. Equation 31 becomes

$$4\psi \frac{\partial^2 W}{\partial \psi^2} + 4 \frac{\partial W}{\partial \psi} - \frac{W}{4} = 0 \tag{33}$$

Equation 33 can be seen to be a Bessel differential equation [5]. The solution to Eq. 33 can be seen to be

$$W = c_1 I_0\left(1/2\sqrt{\tau^2 - X^2}\right) + c_2 K_0\left(1/2\sqrt{\tau^2 - X^2}\right) \tag{34}$$

It can be seen that at the wave front, i.e., $\psi = 0$, W is finite and, therefore, $c_2 = 0$. Far from the wave front and close to the surface, the boundary condition can be written as

$$X = 0, \quad u = 1, \quad \text{or} \quad W = c_1 exp(\tau/2) \tag{35}$$

As ψ is a spatio-temporal variable, the constants of integration c_1 can be a function of either space or time. Applying the boundary condition at the surface, c_1 can be eliminated between Eqs. 34 and 34 to yield in the open interval, $\tau > X$,

$$u = \frac{I_0\left(\frac{\sqrt{\tau^2 - X^2}}{2}\right)}{I_0(\tau/2)} \tag{36}$$

In the domain, $X > \tau$, it can be shown [5] that the solution for the dimensional temperature by a similar approach as above is

$$u = \frac{J_0\left(\frac{\sqrt{X^2 - \tau^2}}{2}\right)}{I_0\,(\tau/2)} \tag{37}$$

At the wave front, $\psi = 0$, Eq. 33 can be solved and

$$\ln(W) = \frac{\psi}{16} \quad \text{or} \quad W = c_3 \exp\left(\frac{\psi}{16}\right)$$

The temperature at the wave front is thus, $u = c_3\exp(-\tau/2) = c_3\exp(-X/2)$. From the boundary condition at $X = 0$, $c_3 = 1.0$. Thus, at the wave front,

$$u = \exp\left(\frac{-X}{2}\right) \tag{38}$$

From Eq. 37 the inertial lag time associated with an interior point in the semi-infinite medium can be calculated by realizing that the first zero of the Bessel function, $J_0(\psi)$, occurs at $\psi = 2.4048$. Thus,

$$4\left(2.4048^2\right) = \frac{x_{\mathrm{p}}^2}{\alpha\tau_{\mathrm{r}}} - \frac{t_{\mathrm{lag}}^2}{\tau_{\mathrm{r}}^2} \tag{39}$$

$$t_{\mathrm{lag}} = \sqrt{x_p^2\frac{\tau_{\mathrm{r}}}{\alpha} - 23.132\tau_{\mathrm{r}}^2} \tag{40}$$

The penetration distance for a given time instant can be developed at the first zero of the Bessel function. Beyond this point, the interior temperatures can be no less than the initial temperature. Thus,

$$X_{\mathrm{pen}} = \sqrt{23.132 + \tau^2} \tag{41}$$

Thus four distinct regimes can be recognized in the solution for the transient temperature at a point $\underline{X}$ in the medium from the surface

(i) $\tau \leq \tau_{\text{lag}}, \ u = 0$ where

$$\tau_{\text{lag}} = \sqrt{X^2 - 23.132} \tag{42}$$

(ii) in the open interval, $\tau_{\text{lag}} < \tau < X$

$$u = \frac{J_0\left(\frac{\sqrt{X^2-\tau^2}}{2}\right)}{I_0\left(\tau/2\right)} \tag{43}$$

(iii) at the wave front, $\tau = X$,

$$u = \exp\left(\frac{-X}{2}\right) \tag{44}$$

(iv) in the open interval, $\tau > X$,

$$u = \frac{I_0\left(\frac{\sqrt{\tau^2-X^2}}{2}\right)}{I_0\left(\tau/2\right)} \tag{45}$$

3 Discussion

The surface heat flux for a semi-infinite medium subject to a constant wall temperature solved by the Fourier parabolic heat conduction model and the hyperbolic damped wave conduction and relaxation model is compared with each other using a MS Excel spreadsheet. Equations 6 and 16 are shown side by side in Fig. 2. The "blow-up" in the Fourier model can be seen at short times. The hyperbolic model is well bounded at short times and reached an asymptotic limit of $q^* = 1$ instead of $q^* = \infty$. There appears to be a crossover at $\tau = 1/2$. It was found that for $\tau > 3.8$ the prediction of

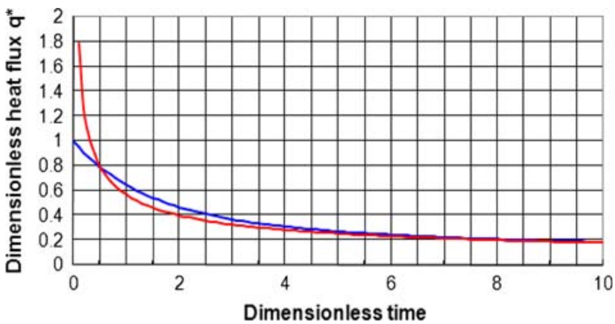

Fig. 2 Comparison of surface flux from the Fourier parabolic heat conduction and hyperbolic damped wave conduction and relaxation models

the hyperbolic model is within 10 % of the parabolic model. It can be seen from Fig. 2 that at large times the predictions of the parabolic and hyperbolic models are the same. For short times, both qualitatively and quantitatively the predictions of the parabolic and hyperbolic models are substantially different.

It is not clear what happens at $\tau = 1/2$. The hyperbolic governing equation can be transformed using the Boltzmann transformation as follows. Let $\tau = X/\sqrt{\tau}$. Equation 7 becomes

$$-\left(2\gamma\frac{\partial u}{\partial \gamma} + \frac{\partial^2 u}{\partial \gamma^2}\right) = \frac{1}{\tau}\left(\gamma\frac{\partial u}{\partial \gamma} - \gamma^2\frac{\partial^2 u}{\partial \gamma^2}\right) \quad (46)$$

For long times, such as $\tau > 1/2$, the RHS of Eq. 46 can be dropped and the LHS solved to yield the solution that is identical with the solution of the Fourier parabolic heat conduction equation, i.e.,

$$u = 1 - \mathrm{erf}\left(\frac{X}{\sqrt{4\tau}}\right) \quad (47)$$

When differentiated and the expression for the flux obtained at the surface and $X = 0$, it can be seen that both the parabolic and the hyperbolic heat conduction equations predict the same reduction of heat flux with time for large times. This is why that beyond $\tau > 1/2$ the predictions of parabolic and hyperbolic models are close to each other as seen in Fig. 2. For short times, $\tau < 1/2$, the microscale time effects becomes important, and when neglected, give rise to a singularity as can be seen from Fig. 2. So the hyperbolic heat conduction model needs to be used for short-time transient applications.

The temperature solution obtained after the Chebyshev polynomial approximation for the integrand in the Baumeister and Hamill solution (Eq. 14) and further integration is shown in Fig. 3. The conditions selected were typical ($\tau = 5$), and Eq. 26 was plotted using an MS Excel spreadsheet. This is shown in Fig. 3. The expression for temperature developed using the method of relativistic transformation (Sharma [4]) for the same condition of $\tau = 5$ is also shown side by side in Fig. 3. It can be seen that both Baumeister and Hamill solution and solution from the relativistic transformation are close to each other, within an average of 12 % deviation from each other. It can also be seen that close to the surface or far from the wave front the numerical errors expected from the Chebyshev polynomial approximation is large. For such conditions the expression developed by the method of relativistic transformation may be used. For conditions close to the wave front, the further integrated expression developed in this study may be used. It is inconclusive whether the Baumeister and Hamill solution violates the second law of thermodynamics close to the surface or it is caused by the numerical errors of integration. The *penetration dimensionless distance* for $\tau = 5$ beyond which there is expected no heat transfer is given by Eq. 41 and is 6.94 by the method of relativistic transformation. Baumeister and Hamill solution predicts a sharp discontinuity past the wave front as shown in Fig. 3. Both the solutions for the transient temperature for the damped wave conduction and relaxation hyperbolic equation from the method of Laplace transforms and Chebyshev economization and

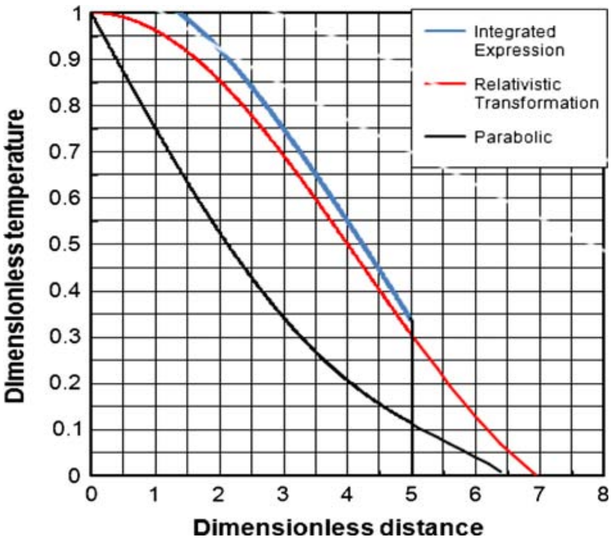

Fig. 3 Temperature distribution in semi-infinite medium by damped wave conduction and relaxation, $\tau = 5$, and parabolic Fourier heat conduction

the method of relativistic transformation are compared against the prediction for the transient temperature by the Fourier parabolic heat conduction model. The transient temperature from Chebyshev economization was found to be within 25 % of the error function solution for the parabolic Fourier heat conduction model. The hyperbolic model solutions compare well with the Fourier model solution for the transient temperature close to the wave front and close to the surface (to within 15 % of each other). The deviations are at the intermediate values.

4 Conclusions

The parabolic Fourier model and hyperbolic model for transient heat flux at the surface for the problem of transient heat conduction in a semi-infinite medium subject to a constant surface temperature boundary condition were found to be within 10 % of each other for times $t > 2\tau_r$ (Fig. 2). This checks out with the Boltzmann transformation and the hyperbolic governing equation reverts to parabolic at long times. At short times there is a "blow-up" in the parabolic model. In the hyperbolic model there is no singularity. This has significant implications in several industrial applications such as fluidized bed heat transfer, CPU overheating, gel acrylamide electrophoresis, etc.

The solution developed by Baumeister and Hamill [44] by the method of Laplace transforms (Eq. 12) was further integrated into a useful expression. A Chebyshev polynomial approximation was used to approximate the integrand with a modified Bessel composite function of space and time of the first kind and first order. The error involved in Chebyshev economization was $4.1 \times 10^{-5}\eta\xi$. The useful expression for the transient temperature was shown in Fig. 3 for a typical time of $\tau = 5$. The dimensionless temperature as a function of dimensionless distance is shown in Fig. 3. The predictions from Baumeister and Hamill and the solution by the method of relativistic transformation are within 12 % of each other on the average. Close to the wave front, the error in the Chebyshev economization is expected to be small and verified accordingly. Close to the surface, the numerical error involved in the Chebyshev economization can be expected to be significant. This can be seen in Fig. 3 close to the surface. The method of relativistic transformation yields bounded solutions without any singularities. The transformation variable, ψ, is symmetric with respect to space and time. It transforms the PDE that governs the wave temperature into a Bessel differential equation. The penetration distance beyond which there is no effect of the step change in temperature at the surface for a considered instant in time is shown in Fig. 3. The solutions from the relativistic transformation of coordinates is an improvement over the Baumeister and Hamill solution and parabolic Fourier solution in depicting the transient heat events in a semi-infinite medium subject to a step-change in boundary temperature. Four regimes in the transient temperature solution for the hyperbolic governing equation using the method of relativistic transformation of coordinates are recognized, and closed form analytical solutions in each regime are given without any singularities. The transient temperature is also found to be consistent with the second law of thermodynamics in all four regimes.

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